STOCHASTIC DIFFERENTIAL EQUATIONS INVOLVING POSITIVE NOISE

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§1. Introduction and motivation

An ordinary stochastic differential may be viewed as a differential equation where some of the coefficients are subject to white noise perturbations. Perhaps the most celebrated example is the equation

$$\frac{dX_t}{dt} = (r + \alpha \cdot W_t)X_t; \quad X_0 = x, \tag{1.1}$$

which is the mathematical model for, say, a population X_t whose relative growth rate has the form $r + \alpha W_t$, where r, α are constants and W_t denotes white noise, which represents random fluctuations due to changes in the environment of the population.

Quite often, however, the nature of the noise is not "white" but biased in some sense. For example, if X_t represents the size/weight of a tree in a random environment then a suitable stochastic differential equation for X_t would be

$$\frac{dX_t}{dt} = (r + \alpha \cdot N_t)X_t; \quad X_0 = x \tag{1.2}$$

where N_t is a stochastic process representing "positive noise" in some sense.

As a second example, consider incompressible fluid flow in a porous medium. Combining Darcy's law and the continuity equation we end up with the partial differential equation

(in x for each t)

$$(1.3)\qquad \begin{cases} div(k(x) \bigtriangledown p(x,t)) = -f(x,t) & \text{for } x \in D_t \\ p(x,t) = 0 & \text{for } x \in \partial D_t \end{cases}$$

$k(x) \geq 0$ is the *permeability* of the medium at the point $x \in \mathbb{R}^3, f(x,t)$ is the given source/sink rate of the fluid at the point x and at time t and $p(x,t)$ is the (unknown) pressure of the fluid, while D_t denotes the set of points x where the fluid has obtained the maximal degree of saturation at time t. (See e.g. Øksendal (1990) for details). Because the permeability $k(x)$ is hard to measure and varies rapidly from point to point we find it natural to propose a stochastic mathematical model where $k(x)$ is regarded as a 3-parameter positive noise process.

There are of course several ways to interpret "positive noise process". For example, many papers have been written about (1.3) with the assumption that $k(x)$ is some ordinary stochastic process parametrized by x, i.e. $k(x) = k(x,w); w \in \Omega$, assuming only nonnegative values.

In particular, in Dikow and Hornung (1987), the following situation is studied:

$$(1.4)\qquad k(x) = k_0(x) \cdot exp\xi(x,w),$$

where $k_0(\cdot), \xi(\cdot,w)$ are (uniformly) Hölder α-continuous for each w, ξ is a bounded process and k_0 is bounded and bounded away from 0. Actually, under these assumptions the operator in (1.3) becomes uniformly elliptic and one can approach the corresponding boundary value problem pathwise (i.e. for each w) by known deterministic methods.

In Øksendal (1990), it is shown how to solve (1.3) for more general types of k: It suffices that $k(x)$ is an A_2-weight in the sense of Muckenhoupt, i.e. that

$$(1.5)\qquad \sup_B\left(\frac{1}{|B|}\int_B k(x)dx\right)\left(\frac{1}{|B|}\int_B \frac{1}{k(x)}dx\right) < \infty$$

the sup being taken over all balls $B \subset \mathbb{R}^3, dx$ denoting Lebesgue measure and $|B| = \int_B dx$ being the volume of B. In particular, this allows k to have zeroes.

Here we propose a stochastic approach which is radically different from the methods mentioned above:

We represent the stochastic quantities involved by a suitable *functional of multi-parameter white noise* $W_x; x \in \mathbb{R}^n$. For example, in equation (1.3) we put $k(x)$ equal to some positive functional white noise (see §7). Then we look for a solution of the corresponding stochastic partial differential equation in some generalized/distribution sense. Finally explicit information about the solution can then be obtained by taking averages of the distribution solution. The philosophy behind this approach is in a sense similar to the philosophy behind ordinary stochastic differential equations involving white noise: It is better to calculate/solve the equation first and then take averages rather than take the average first and then calculate.

In this report we explain the first step in such a program: Here we will consider only 1-parameter equations, the multi-parameter case will be discussed in future papers.

To summarize, the purpose of this paper is to construct a (1-parameter) noise concept which is general enough to include good models for positive noise from real life situations, yet at the same time allows us to use calculus to solve differential equations involving this noise.

The key concept in our approach is the *functional process*, a distribution valued process which we construct in §4, after giving some background in §2-3. In §5 we introduce the Hermite transform $\mathcal{H}$, which associates to each functional process a (deterministic) complex valued function on $\mathbb{C}_0^{\mathbb{N}} = \{(z_1, z_2, \cdots); z_j \in \mathbb{C}, \exists N \text{ with } z_j = 0 \text{ for } j > N\}$. This transform is fundamental for our subsequent calculus on functional processes.

Even though a functional process is distribution valued we show in §6 that one may define a functional calculus on such processes, in the sense that one can compose them with analytic functions which do not grow too fast at ∞.

In §7 we introduce the concept of a *positive* functional process and we prove that the (Wick) product of two positive processes is again a positive process. This justifies the use of this concept to model non-negative growth in a random environment.

Finally, in §8 we illustrate our method by solving the 1-dimensional version of (1.3). We also give an estimate for the (mean square) error that we make if we replace the noisy equation (1.3) by the equation where k is replaced by its (constant) mean value.

Our main inspiration has come from the works of Ito (1951), Hida (1980) and Kuo (1983), but it has gradually become clear to us that many other authors have discussed more or less related subjects, for example in connection with mathematical physics (renormalization etc.). We have included those that we know about in the reference list, and apologize to those that should have been mentioned that we are not yet aware of.

§2. The white noise probability space

Since white noise is so fundamental for our construction, we recall some basic facts about this generalized (i.e. distribution valued) process:

For $n = 1, 2, \cdots$ let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all rapidly decreasing smooth (C^∞) functions on $\mathbb{R}^n$. Then $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space under the family of seminorms

$$\|f\|_{N,\alpha} = \sup_{x \in \mathbb{R}} (1 + |x|^N) |\partial^\alpha f(x)|,$$

where $N \geq 0$ is an integer and $\alpha = (\alpha_1, \cdots, \alpha_k)$ is a multi-index of non-negative integers

α_j. The space of *tempered distributions* is the dual $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$, equipped with the weak star topology.

Now let $n = 1$ for the rest of this section and put $\mathcal{S} = \mathcal{S}(\mathbb{R}), \mathcal{S}' = \mathcal{S}'(\mathbb{R})$. By the Bochner-Minlos theorem (see e.g. Gelfand and Vilenkin (1964)) there exists a probability measure μ on $(\mathcal{S}', \mathcal{B})$ (where $\mathcal{B} = \mathcal{B}(\mathcal{S}')$ denotes the Borel subsets of $\mathcal{S}'$) such that

$$E^\mu[e^{i<\omega,\phi>}] := \int_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \text{ for all } \phi \in \mathcal{S}, \tag{2.1}$$

where $\|\phi\|^2 = \|\phi\|^2_{L^2(\mathbb{R})}$ and $<\omega, \phi> = \omega(\phi)$ for $\omega \in \mathcal{S}'$. It follows from (2.1) that

$$\int_{\mathcal{S}'} f(<\omega,\phi>) d\mu(\omega) = (2\pi\|\phi\|^2)^{-\frac{1}{2}} \int_{\mathbb{R}} f(t) e^{-\frac{t^2}{2\|\phi\|^2}} dt; \phi \in \mathcal{S}, \tag{2.2}$$

for all f such that the integral on the right converges (It suffices to prove (2.2) for $f \in C_0^\infty(\mathbb{R})$, i.e. f smooth with compact support. Such a function f is the inverse Fourier transform of its Fourier transform $\hat{f}$ and we obtain (2.2) by (2.1) and the Fubini theorem). In particular, if we choose $f(t) = t^2$ we get from (2.2)

$$E^\mu[<\omega,\phi>^2] = \|\phi\|^2; \phi \in \mathcal{S}. \tag{2.3}$$

This allows us to extend the definition of $<\omega,\phi>$ from $\phi \in \mathcal{S}$ to $\phi \in L^2(\mathbb{R})$ for a.a. $\omega \in \mathcal{S}'$, as follows:

$$<\omega,\phi> := \lim_{k\to\infty} <\omega,\phi_k> \text{ for } \phi \in L^2(\mathbb{R}), \tag{2.4}$$

where ϕ_k is any sequence in $\mathcal{S}$ such that $\phi_k \to \phi$ in $L^2(\mathbb{R})$ and the limit in (2.4) is in $L^2(\mathcal{S}', \mu)$.

In particular, if we define

$$\tilde{B}_t(\omega) := <\omega, \chi[0,t]> \tag{2.5}$$

then we see that $(\tilde{B}_t, \mathcal{S}', \mu)$ becomes a Gaussian process with mean 0 and covariance

$$E^\mu[\tilde{B}_t(\omega)\tilde{B}_s(\omega)] = \int_{\mathcal{S}'} <\omega, \chi_{[0,t]}> \cdot <\omega, \chi_{[0,s]}> d\mu(\omega)$$

$$= \int_{\mathbb{R}} \chi_{[0,t]}(x) \cdot \chi_{[0,s]}(x) dx = min(s,t), \text{using (2.3)}.$$

Therefore $\tilde{B}_t$ is essentially a Brownian motion, in the sense that there exists a t-continuous version B_t of $\tilde{B}_t$:

$$\mu(\{\omega; B_t(\omega) = \tilde{B}_t(\omega)\}) = 1 \quad \text{for all } t.$$

If $u \in L^2(\mathbb{R})$ we define, using (2.4)

$$\int_{-\infty}^{\infty} \phi(t) dB_t(\omega) = <\omega, \phi> \tag{2.5}$$

which coincides with the classical Ito integral if supp$\phi \subset [0, \infty)$.

If we define *the white noise process* W_ϕ by

$$W_\phi(\omega) = <\omega, \phi> \quad \text{for } \phi \in \mathcal{S}, \omega \in \mathcal{S}' \tag{2.6}$$

then W_ϕ may be regarded as the distributional derivative of B_t, in the sense that, if $\phi \in \mathcal{S}$

$$\begin{aligned} <\frac{d}{dt} B_t(\omega), \phi> &= -\int_{-\infty}^{\infty} \phi'(t) B_t(\omega) dt = \int_{-\infty}^{\infty} \phi(t) dB_t(\omega) \\ &= \lim_{\Delta t_j \to 0} \sum_j \phi(t_j)(B_{t_{j+1}} - B_{t_j}) = \lim_{\Delta t_j \to 0} \sum_j \phi(t_j) <\omega, \chi_{(t_j, t_j+1]}> \\ &= \lim_{\Delta t_j \to 0} <\omega, \sum_j \phi(t_j) \chi_{(t_j, t_j+1]}> = <\omega, \phi> = W_\phi(\omega), \end{aligned}$$

where the second identity is based on integration by parts for Ito integrals.

§3. Generalized white noise functionals

In this section we summarize Hida's concept of generalized white noise functionals. They will be a natural starting point for our definition of functional processes in §4. For more details see Hida (1980) or Hida-Kuo-Potthoff-Streit (1991).

By the Wiener-Ito chaos theorem (Ito (1951)), we can write any function $f \in L^2(\mu)$ $(= L^2(\mathcal{S}', \mu))$ on the form

$$f = \sum_{n=0}^{\infty} \int f_n dB^{\otimes n}, \tag{3.1}$$

where

$$f_n \in \hat{L^2}(\mathbb{R}^n, dx), \tag{3.2}$$

i.e. $f_n \in L^2(\mathbb{R}^n, dx)$ and f_n is *symmetric* (in the sense that $f_n(x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}) = f(x_1, \cdots, x_n)$ for all permutations σ of $(1, 2, \cdots, n)$) and

$$\begin{aligned} \int f_n dB^{\otimes n} &= \int_{\mathbb{R}^n} f_n(u) dB_u^{\otimes n} \\ &= n! \int_{-\infty}^{\infty} (\int_{-\infty}^{u_n} \cdots \int_{-\infty}^{u_3} (\int_{-\infty}^{u_2} f_n(u_1, \cdots, u_n) dB_{u_1}) dB_{u_2} \cdots dB_{u_{n-1}}) dB_{u_n} \end{aligned} \tag{3.3}$$

for $n \geq 1$, while the $n = 0$ term in (3.1) is just a constant f_0.

For a general (non-symmetric) $f \in L^2(\mathbb{R}^n)$ we define

$$\int f dB^{\otimes n} := \int \hat{f} dB^{\otimes n} \tag{3.4}$$

where $\hat{f}$ is the symmetrization of f, defined by

$$\hat{f}(u_1, \cdots, u_n) = \frac{1}{n!} \sum_{\sigma} f(u_{\sigma_1}, \cdots, u_{\sigma_n}), \tag{3.5}$$

the sum being taken over all permutations σ of $(1, 2, \cdots, n)$.

With f, f_n as in (3.1) we have

$$\|f\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2(\mathbb{R}^n)} \tag{3.6}$$

Note that (3.6) follows from (3.1) and (3.3) by the Ito isometry, since

$$E[(\int_{\mathbb{R}^n} f_n dB^{\otimes n})(\int_{\mathbb{R}^m} f_m dB^{\otimes m})] = 0 \quad \text{for } n \neq m$$

and

$$E[(\int_{\mathbb{R}^n} f_n dB^{\otimes n})^2] = (n!)^2 E[(\int_{-\infty}^{\infty} \cdots (\int_{-\infty}^{u_2} f_n(u_1, \cdots, u_n) dB_1) \cdots dB_{u_n})^2]$$

$$= (n!)^2 \cdot \int_{-\infty}^{\infty} \cdots (\int_{-\infty}^{u_2} f_n^2(u_1, \cdots, u_n) du_1) \cdots du_n = n! \int_{\mathbb{R}^n} f_n^2 dx$$

Here $B_u(\omega); u \geq 0, \omega \in \mathcal{S}'$ is the 1-dimensional Brownian motion associated with the white noise probability space $(\mathcal{S}', \mu)$ as explained in §2.

For $s \in \mathbb{R}$ we define the Sobolev space $H^s = H^s(\mathbb{R}^n)$ by

$$H^s = \{\psi \in \mathcal{S}'(\mathbb{R}^n); \|\psi\|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\hat{\psi}(y)|^2 (1 + |y|^2)^s dy < \infty\}, \tag{3.7}$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. Then the dual of H^s is simply H^{-s}, for all $s \in \mathbb{R}$.

The *Hida test function space* $(L^2)^+ = (L^2)^+(\mu)$ is the subspace of $L^2(\mu)$ consisting of all functions $f \in L^2(\mu)$ of the form

$$f = \sum_{n=0}^{\infty} \int f_n dB^{\otimes n} \tag{3.8}$$

where $f_n \in \hat{H}^{\frac{n+1}{2}}(\mathbb{R}^n)$ and

(3.9)
$$\|f\|^2_{(L^2)^+} := \sum_{n=0}^{\infty} n!\|f_n\|^2_{\hat{H}^{\frac{n+1}{2}}(\mathbb{R}^n)},$$

$\hat{H}^{\frac{n+1}{2}}(\mathbb{R}^n)$ being the space of symmetric functions in the Sobolev space $H^{\frac{n+1}{2}}(\mathbb{R}^n)$. Note that by the Sobolev imbedding theorem each f_n has a continuous version, so we may - and will - assume from now on that each f_n is continuous.

The *Hida space* $(L^2)^-$ *of generalized white noise functionals* (see Hida (1980), Kuo (1983)) is the dual of the space $(L^2)^+$. Formally we may represent an element F of $(L^2)^-$ as follows

(3.10)
$$F = \sum_{n=0}^{\infty} \int F_n dB^{\otimes n},$$

where

(3.11)
$$F_n \in \hat{H}^{-\frac{n+1}{2}}(\mathbb{R}^n) \text{ for all } n,$$

and

(3.12)
$$\|F\|^2_{(L^2)^-} = \sum_{n=0}^{\infty} n!\|F_n\|^2_{\hat{H}^{-\frac{n+1}{2}}(\mathbb{R}^n)},$$

the action $F(f) =< F, f >$ of F on an element $f \in (L^2)^+$ being given by

(3.13)
$$F(f) = \sum_{n=0}^{\infty} n! F_n(f_n)$$

Note that in the special case when $F \in L^2(\mu)$ then (3.13) coincides with the result of taking the inner product of g and f in $L^2(\mu)$, since, by (3.5) and (3.6),

(3.14)
$$\begin{aligned} E[Ff] &= \sum_{n=0}^{\infty} E[(\int_{\mathbb{R}^n} F_n dB^{\otimes n})(\int_{\mathbb{R}^n} f_n dB^{\otimes n})] = \sum_{n=0}^{\infty} n! \int_{\mathbb{R}^n} F_n(u) f_n(u) du \\ &= \sum_{n=0}^{\infty} n! F_n(f_n) \end{aligned}$$

More generally one can consider the space $(\mathcal{S})^*$ of *generalized white noise functionals* consisting of elements of the form $\sum \int F_n dB^{\otimes n}$ where $F_n \in \mathcal{S}(\mathbb{R}^n)$, equipped with a certain topology. See Hida-Kuo-Potthoff-Streit (1991).

In addition to the natural vector space structure the space $(L^2)^-$ (and $(\mathcal{S})^*$) also has a multiplication called *Wick* multiplication $\diamond$, defined by

(3.15)
$$(\int_{\mathbb{R}^n} F_n dB^{\otimes n}) \diamond (\int_{\mathbb{R}^m} G_m dB^{\otimes m}) = \int_{\mathbb{R}^{n+m}} F_n \hat{\otimes} G_m dB^{\otimes(n+m)},$$

where $F_n \hat{\otimes} G_m$ denotes the symmetrized tensor product of F_n and G_m. (If F_n and G_m are functions, this means that we first form the usual tensor product

$$F_n \otimes G_m(z) = F_n \otimes G_m(x_1, \cdots, x_n, y_1, \cdots, y_m) = F_n(x_1, \cdots, x_n) G_m(y_1, \cdots, y_m)$$

and then symmetrize the result, so that

$$F_n \hat{\otimes} G_m(z) = \frac{1}{(n+m)!} \sum_{\sigma} F_n \otimes G_m(z_{\sigma_1}, \cdots, z_{\sigma_{n+m}}),$$

the sum being taken over all permulations σ of $(1, \cdots, n+m)$. This definition extends in the usual way to tempered distributions F_n, G_m).

Note that if $F_n \in \hat{H}^{-\frac{n+1}{2}}(\mathbb{R}^n), G_m \in \hat{H}^{-\frac{m+1}{2}}(\mathbb{R}^m)$ with $n \leq m$ then

$$F_n \hat{\otimes} G_m \in \hat{H}^{-\frac{m+1}{2}}(\mathbb{R}^{n \times m})$$

and

$$\|F_n \hat{\otimes} G_m\|_{\hat{H}^{-\frac{m+1}{2}}(\mathbb{R}^{n \times m})} \leq \|F_n\|_{\hat{H}^{-\frac{n+1}{2}}(\mathbb{R}^n)} \cdot \|G_m\|_{\hat{H}^{-\frac{m+1}{2}}(\mathbb{R}^m)} \tag{3.16}$$

Therefore we can extend in a natural way the product given in (3.13) to work for more general $F \in (L^2)^-, G \in (L^2)^-$ by defining

$$\begin{aligned} F \diamond G &= \lim_{k \to \infty} \left(\sum_{n=0}^{k} \int F_n dB^{\otimes n}\right) \diamond \left(\sum_{m=0}^{k} \int G_m dB^{\otimes m}\right) \\ &= \sum_{n,m=0}^{\infty} \int F_n \hat{\otimes} G_m dB^{\otimes(n+m)}. \end{aligned} \tag{3.17}$$

In view of (3.16) we see that $F \diamond G \in (L^2)^-$ if (for example)

$$\sum_{k=0}^{\infty} k \cdot k! \sum_{n+m=k} \|F_n\|^2_{\hat{H}^{-\frac{n+1}{2}}(\mathbb{R}^n)} \cdot \|G_m\|^2_{\hat{H}^{-\frac{m+1}{2}}(\mathbb{R}^m)} < \infty. \tag{3.18}$$

Remarks

1) In the literature the symbol $: FG :$ is often used for the Wick product $F \diamond G$. For more details on the connection between :: and $\diamond$ see the remark following Theorem 5.1.
2) Several motivations exist for this definition of multiplication of functional processes. We give some of them here:

a) First note that multiplication of a deterministic quantity with a random one reduces to the usual multiplication. Going one step further, one might say that the Wick product can be regarded as the product one would obtain if the two stochastic factors were independent. In general our quantities are of course dependent, but in a sense we ignore this when we form the Wick product. This point of view is related to the situation in thermodynamics for example, where technically the path of a single particle of course depends on the paths of the others, but the connection is so weak/chaotic that it is not unreasonable to assume that the effect of it averages to zero in time. The Wick product may be viewed as an axiomatic way of stating such a principle. Moreover, this special product has the technical advantage of preserving the martingale property. And - as we shall see later (§9) - the product provides a new approach to the Ito calculus.

b) Wick multiplication can also be motivated from a renormalization point of view: Arguing heuristically we see that white noise W_t may be represented as an element of $(L^2)^-$ as follows

$$W_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} dB_u = \int_{\mathbb{R}} \delta_t(u) dB_u,$$

where $\delta_t \in H^{-1}(\mathbb{R})$ is the Dirac measure at t.

Therefore one should have, by Ito's formula,

$$W_t^2 \approx (\frac{1}{\Delta t} \int_t^{t+\Delta t} dB_u)^2 = \frac{2}{(\Delta t)^2} \int_t^{t+\Delta t} (\int_t^v dB_u) dB_v) + \frac{1}{\Delta t}$$

The additive renormalization of this is

$$2 \int_t^{t+\Delta t} (\int_t^v \frac{dB_u}{\Delta t}) \frac{dB_v}{\Delta t} \to \int \int \delta_t(u) \delta_t(v) dB_u dB_v \text{ as } \Delta t \to 0$$

which motivates the definition

$$\int \delta_t dB \diamond \int \delta_t dB = \int \delta_t \otimes \delta_t dB^{\otimes 2},$$

in agreement with (3.15).

3) It should be pointed out that the argument above, as well as the notation

$$\text{(3.19)} \qquad \int_{\mathbb{R}^n} F_n dB^{\otimes n} \quad \text{for} \quad F_n \in \hat{H}^{-\frac{n+1}{2}}(\mathbb{R}^n)$$

from the general representation (3.10) of elements in $(L^2)^-$, is just formal: (3.19) is not defined as an Ito integral. Nevertheless, (3.19) - slightly modified and generalized - makes sense as a distribution valued process and this is the starting point for the type of *generalized white noise functional processes* (*functional processes*, for short) which we now construct.

§4. Functional processes

Let $s \in \mathbb{R}$ and suppose $F \in H^{-s}(\mathbb{R}^n; L^2(\mathbb{R}^n))$ is an L^2-valued H^{-s}-distribution, i.e. F is a linear map from $H^s(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ such that there exists $K < \infty$ with

$$\| < F, \phi > \|_{L^2} \leq K \cdot \|\phi\|_{H^s} \text{ for all } \phi \in H^s \tag{4.1}$$

where $< F, \phi > := F\phi := F(\phi)$.

Then

$$Y\phi(\omega) = \int_{\mathbb{R}^n} F\phi(u) dB_u^{\otimes n}(\omega) \tag{4.2}$$

is defined for $a.a.\omega$ as an $L^2(\mathcal{S}')$-limit in the usual way for each $\phi \in \mathcal{S} = \mathcal{S}(\mathbb{R}^n)$. Thus Y is a *random linear functional* on H^s, in the sense that

$$Y(c_1\phi_1 + c_2\phi_2)(\omega) = c_1 Y\phi_1(\omega) + c_2 Y\phi_2(\omega) \text{ for } a.a.\omega,$$

for all $\phi_1, \phi_2 \in H^s$ and all $c_1, c_2 \in \mathbb{R}$.

It follows that Y *has a version with values in* H^{-r}, if

$$r > s + \frac{n}{2}.$$

See Walsh (1984), theorem 4.1. In the following we will assume that this H^{-r} version is chosen, so that we may regard

$$Y(\cdot, \omega) = \int_{\mathbb{R}^n} F(\cdot) dB^{\otimes n}(\omega) \text{ with } F \in H^{-s}(\mathbb{R}^n; L^2(\mathbb{R}^n)) \tag{4.3}$$

as an $H^{-r}(\mathbb{R}^n)$-valued stochastic process.

For notational simplicity we put

$$H^{-\infty} = \bigcup_{k=1}^{\infty} H^{-k} \tag{4.4}$$

so that if $F \in H^{-\infty}$ then $F \in H^{-k}$ for some k and then we write

$$\|F\|_{H^{-\infty}} = \|F\|_{H^{-k}}.$$

DEFINITION 4.1. A *functional process* $\{X(\cdot,\omega)\}_{\omega\in\mathcal{S}'}$ is a sum of distribution valued processes of the form

$$X_\phi(\omega) = X(\phi,\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} F^{(n)}(\phi^{\otimes n}) dB^{\otimes n}(\omega); \phi \in \mathcal{S}, \omega \in \mathcal{S}' \tag{4.5}$$

where

$$F^{(n)}(\cdot) \in H^{-\infty}(\mathbb{R}^n; L^2(\mathbb{R}^n)) \text{ for all } n \geq 1$$

and

$$F^{(0)}(\cdot) \in H^{-\infty}(\mathbb{R}).$$

Moreover, we assume that

$$E[|X(\phi,\omega)|^2] = \sum_{n=0}^{\infty} n! \int_{\mathbb{R}^n} < F^{(n)}, \phi^{\otimes n} >^2 (u) du < \infty \tag{4.6}$$

for all $\phi \in \mathcal{S}$ with $\|\phi\|_{L^2}$ sufficiently small.

To make the notation more suggestive we often write the functional process $X(\phi,\omega)$ on the form

$$X_t(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} F^{(n)}_{t,\cdots,t}(u) dB^{\otimes n}_u(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} F^{(n)}_t dB^{\otimes n}, \tag{4.7}$$

where each $F^{(n)}_t(u)$ is really an L^2-valued distribution in the t-variable, $t = (t_1,\cdots,t_n)$. The distributional derivative of X_t with respect to t is then defined by

$$\frac{dX_t}{dt}(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \frac{d}{dt} F^{(n)}_{t,\cdots,t}(u) dB^{\otimes n}_u(\omega) \tag{4.8}$$

where

$$\frac{d}{dt} F^{(n)}_{t,t,\cdots,t} = \left(\sum_{j=1}^{n} \frac{\partial F^{(n)}}{\partial x_j}\right)_{x=(t,\cdots,t)},$$

$\frac{\partial}{\partial x_j}$ denoting the usual distributional derivative with respect to x_j, i.e.

$$< \frac{\partial F^{(n)}}{\partial x_j}, \psi > = - < F^{(n)}, \frac{\partial \psi}{\partial x_j} > \text{ for } \psi = \psi(x_1,\cdots,x_n) \in \mathcal{S}(\mathbb{R}^n).$$

EXAMPLE 4.2. The *white noise process* W_t can be represented as a functional process as follows:

$$W_t = \int_{-\infty}^{\infty} \delta_t(u) dB_u \tag{4.9}$$

where $\delta_t(u)$ is the usual Dirac measure, i.e.

$$< \delta_t(u), \phi(t) >= \phi(u)$$

To see this note that, according to the definition above, (4.9) means that

$$W_\phi(\omega) = \int \phi(u) dB_u(\omega) \tag{4.10}$$

which is just a reformulation of (2.6).

EXAMPLE 4.3. By the previous example together with (3.15) we can represent *the square of white noise* as follows:

$$W_t^{\diamond 2} = W_t \diamond W_t = \int_{\mathbb{R}^2} \delta_t(u)\delta_t(v) dB_u dB_v, \tag{4.11}$$

This means that as a distribution valued process $W_t^{\diamond 2}$ can be written

$$W^{\diamond 2}(\phi \otimes \psi, \omega) = \int_{\mathbb{R}^2} \phi(u)\psi(v) dB_u dB_v(\omega); \phi, \psi \in \mathcal{S}$$

In particular,

$$W^{\diamond 2}(\phi \otimes \phi, \omega) = \int_{\mathbb{R}^2} \phi(u)\phi(v) dB_u dB_v(\omega); \phi \in \mathcal{S}$$

so a more correct notation for $W_t^{\diamond 2}$ would be $W_{t,t}^{\diamond 2}$.

EXAMPLE 4.4. The functional process

$$X_t(\omega) = \int 1_t(u) 1_t(v) dB_u dB_v = B_t^{\diamond 2}(\omega)$$

where

$$1_t(u) = \begin{cases} 0 & \text{if} \quad u > t \\ 1 & \text{if} \quad u \leq t \end{cases}$$

has a derivative equal to

$$\begin{aligned} \frac{dX_t}{dt} &= \int (1_t(u)\delta_t(v) + \delta_t(u)1_t(v)) dB_u dB_v \\ &= 2\int 1_t \hat{\otimes} \delta_t dB^{\otimes 2} = 2(\int 1_t dB) \diamond (\int \delta_t dB) \\ &= 2B_t \diamond W_t, \end{aligned}$$

which is in agreement with *usual* (not Ito) differentiation rules.

§5. The Hermite transform $\mathcal{H}$

We now show that to a given functional process X_ϕ we can associate an (a.e. defined) function

$$\mathcal{H}(X_\phi) := \tilde{X}_\phi : \mathbb{C}_0^{\mathbb{N}} \to \mathbb{C}$$

where $\mathbb{C} = \{x+iy; x, y \in \mathbb{R}\}$ denotes the set of complex numbers and $\mathbb{C}_0^{\mathbb{N}} = \{(z_1, z_2, \cdots,); z_j \in \mathbb{C}, \exists N \text{ with } z_j = 0 \text{ for } \forall j > N\}$. This connection will be fundamental for the rest of this paper.

Fix an orthonormal family $\{\zeta_k\}_{k=1}^{\infty}$ in $L^2(\mathbb{R})$. Let

$$X_\phi = \sum_{n=0}^{\infty} \int F_\phi^{(n)}(u) dB^{\otimes n} \tag{5.1}$$

be a functional process, and keep $\phi \in \mathcal{S}$ fixed. Using multi-index notation each function $F_\phi^{(n)}(u_1, \cdots, u_n)$ may be (uniquely) written

$$F_\phi^{(n)}(u_1, \cdots, u_n) = \sum_{|\alpha|=n} c_\alpha^{(n)} \zeta^{\otimes\alpha}(u_1, \cdots, u_n) \tag{5.2}$$

for suitable constants $c_\alpha^{(n)} = c_\alpha^{(n)}(\phi)$, where $\alpha = (\alpha_1, \cdots, \alpha_m)$, $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and

$$\zeta^{\otimes\alpha} = \zeta_1^{\otimes\alpha_1} \otimes \zeta_2^{\otimes\alpha_2} \otimes \cdots \otimes \zeta_m^{\otimes\alpha_m}$$

This gives the (unique) representation

$$X_\phi = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha^{(n)} \int \zeta^{\otimes\alpha} dB^{\otimes n} = \sum_\alpha c_\alpha \int \zeta^{\otimes\alpha} dB^{\otimes|\alpha|} \tag{5.3}$$

where $c_\alpha^{(n)}(\cdot) \in H^{-\infty}(\mathbb{R}^n)$ for $n \geq 1, c_\alpha^{(0)}(\cdot) \in H^{-\infty}(\mathbb{R})$.

The iterated Ito integrals on the right of (5.3) can be computed using the Hermite polynomials h_n defined by

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \cdot \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}})$$

The result is the following:

THEOREM 5.1 (Ito (1951), p. 162)

$$\int \zeta_1^{\otimes\alpha_1} \otimes \zeta_2^{\otimes\alpha_2} \otimes \cdots \otimes \zeta_m^{\otimes\alpha_m} dB^{\otimes|\alpha|} = \prod_{j=1}^{m} h_{\alpha_j}(x_j) \tag{5.4}$$

where $x_j = \int \zeta_j(u) dB_u; j = 1, 2, \cdots$

Remark. Using Theorem 5.1 we can now explain more precisely the relation between the multiplication $\diamond$ defined by (3.15) and the classical Wick product:

If X is a real Gaussian random variable with mean 0 and variance $E[X^2] = \sigma^2$ then the k'th Wick power of X is defined by

$$:X^k:= \sigma^k h_k(\frac{X}{\sigma})$$

(See e.g. Simon (1974) p.11).

For example, if $X = \int\limits_{\mathbb{R}} f(t)dB_t$ with $f \in L^2(\mathbb{R})$ then

$$:X^k:= \|f\|^k h_k(\frac{X}{\|f\|}) \text{ where } \|f\| = \|f\|_{L^2}.$$

On the other hand, applying Theorem 5.1 to $\alpha = \alpha_1 = k, \zeta_1 = \|f\|^{-1}f$, we see that

$$X^{\diamond k} = (\|f\| \cdot \int \|f\|^{-1} f dB)^{\diamond k} = \|f\|^k h_k(\frac{X}{\|f\|})$$

Thus

$$:X^k:= X^{\diamond k} \quad \text{for such } X.$$

By polarization we obtain, if $Y = \int g(t)dB$ with $g \in L^2(\mathbb{R})$,
$2:XY:=:(X+Y)^2: - :X^2: - :Y^2:= 2X \diamond Y$.

However, if we more generally consider two random variables U, V of the form

$$U = \int\limits_{\mathbb{R}^n} f_n dB^{\otimes n}, V = \int\limits_{\mathbb{R}^m} g_m dB^{\otimes m} \text{ with } f_n \in L^2(\mathbb{R}^n), g_m \in L^2(\mathbb{R}^m),$$

then by the definition on p.12 in Simon (1974) we have

$$:UV:= UV - E[UV]$$

This is not necessarily the same as $U \diamond V$. For example, if

$$U = \int\limits_{\mathbb{R}^2} f^{\otimes 2} dB^{\otimes 2}, V = \int\limits_{\mathbb{R}} f dB \text{ with } f = \chi_{[0,t]},$$

then $U = V^{\diamond 2}$ so by the above

$$U \diamond V = V^{\diamond 3} = \|f\|^3 h_3(\frac{V}{\|f\|}) = V^3 - 3\|f\|^2 V = B_t^3 - 3tB_t,$$

while

$$:UV := (B_t^2 - t)B_t - E[(B_t^2 - t)B_t] = B_t^3 - tB_t.$$

Nevertheless, we have found the relation between $\diamond$ and : : so close that we feel that it is natural to use the name Wick product for $\diamond$.

The next representation of Hermite polynomials is well known:

$$h_k(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x+iy)^k e^{-\frac{1}{2}y^2} dy \quad k = 1, 2, \cdots \tag{5.5}$$

Define a measure λ on the product σ-algebra on $\mathbb{R}^{\mathbb{N}}$ by

$$\int f(y) d\lambda(y) = \int_{-\infty}^{\infty} \cdots \Big(\int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f(y_1, \cdots, y_n) e^{-\frac{1}{2}y_1^2} \frac{dy_1}{\sqrt{2\pi}} \Big) e^{-\frac{1}{2}y_2^2} \frac{dy_2}{\sqrt{2\pi}} \Big) \cdots e^{-\frac{1}{2}y_n^2} \frac{dy_n}{\sqrt{2\pi}} \tag{5.6}$$

if $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a bounded function depending only on finitely many variables $y_1, \cdots, y_n$. (The formula (5.6) defines λ as a premeasure on the algebra generated by finite products of sets in $\mathbb{R}$ and so λ has a unique extension to the product σ-algebra (Folland (1984)).

Combining (5.3)-(5.6) we obtain

$$\begin{aligned} X_\phi &= \sum_{n=0}^{\infty} \Big(\int \sum_{|\alpha|=n} c_\alpha \zeta^{\otimes \alpha} \Big) dB^{\otimes n} = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha \int \zeta^{\otimes \alpha} dB^{\otimes n} \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha \Big[\int (x+iy)^\alpha d\lambda(y) \Big]_{x=\int \zeta dB} \end{aligned} \tag{5.7}$$

where $(x+iy)^\alpha = (x_1+iy_1)^{\alpha_1}(x_2+iy_2)^{\alpha_2} \cdots (x_m+iy_m)^{\alpha_m}$ if $\alpha = (\alpha_1, \cdots, \alpha_m)$ and we evaluate the right hand integrals at $x_j = \int \zeta_j dB, j = 1, 2, \cdots$.

DEFINITION 5.2 Let $X_\phi = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha \int \zeta^{\otimes \alpha} dB^{\otimes n}$ be a functional process represented as in (5.3). Then the *Hermite transform* (or $\mathcal{H}$-transform) of X_ϕ is the formal power series in infinitely many complex variables $z_1, z_2, \cdots$ given by

$$\mathcal{H}(X_\phi)(z) = \tilde{X}_\phi(z) = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha z^\alpha = \sum_\alpha c_\alpha z^\alpha \tag{5.8}$$

where $z = (z_1, z_2, \cdots)$, $z^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ if $\alpha = (\alpha_1, \cdots, \alpha_m)$ and c_α is defined by (5.3).

Remark. The Hermite transform is related to the ray analytic function associated to the $\mathcal{S}$-transform, namely $z \to \mathcal{S}f(z\phi) := \int_{\mathcal{S}'} f(w + z\phi) d\mu(w)$ for $z \in \mathbb{C}, f \in L^2(\mu), \phi \in \mathcal{S}$. See Hida-Kuo-Potthoff-Streit (1991) for more information about the $\mathcal{S}$-transform.

The sum in (5.8) converges for all $z \in \mathbb{C}_0^{\mathbb{N}}$, where

$$\mathbb{C}_0^{\mathbb{N}} = \{(z_1, z_2, \cdots); \text{ only finitely many of the } z_j's \text{ are nonzero}\}$$

More generally, $\tilde{X}_\phi(z)$ is defined for all $z = (z_1, z_2, \cdots) \in \mathbb{C}^{\mathbb{N}}$ such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{|\alpha|=n} |z^\alpha|^2 < \infty \tag{5.9}$$

In fact, for such z the series in (5.8) is absolutely convergent, since

$$\begin{aligned}\sum_{n=0}^{\infty} \Big(\sum_{|\alpha|=n} |c_\alpha| \cdot |z^\alpha| \Big) &\le \sum_{n=0}^{\infty} \Big(n! \sum_{|\alpha|=n} c_\alpha^2\Big)^{\frac{1}{2}} \Big(\frac{1}{n!} \sum_{|\alpha|=n} |z^\alpha|^2\Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{n=0}^{\infty} n! \sum_{|\alpha|=n} c_\alpha^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{|\alpha|=n} |z^\alpha|^2\Big)^{\frac{1}{2}} < \infty\end{aligned}$$

because

$$\sum_{n=0}^{\infty} n! \sum_{|\alpha|=n} c_\alpha^2 = \sum_{n=0}^{\infty} n! \|F^{(n)}\|^2_{L^2(\mathbb{R}^n)} = E[X_\phi^2] < \infty.$$

In particular, if $z = (z_1, \cdots, z_N, 0, 0, \cdots)$ and $|z_k| \le M$ for all k then

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{|\alpha|=n} |z^\alpha|^2 \le \sum_{n=0}^{\infty} \frac{1}{n!} M^{2n} \binom{N+n-1}{n} \le \sum_{n=0}^{\infty} \frac{M^{2n} \cdot n^N}{n!} < \infty,$$

so (5.8) converges absolutely. Therefore, if we for all N put

$$z^{(N)} := (z_1, \cdots, z_N, 0, 0, \cdots)) \text{ if } z = (z_1, \cdots, z_N, z_{N+1}, \cdots) \in \mathbb{C}^{\mathbb{N}}$$

and define

$$\tilde{X}_\phi^{(N)}(z) = \tilde{X}_\phi(z^{(N)}) \tag{5.10}$$

then the power series for $\tilde{X}_\phi^{(N)}$ converges uniformly on compacts in the variables $z_1, \cdots, z_N$. Hence we have proved:

LEMMA 5.3 $\tilde{X}_\phi^{(N)}$ is an analytic function in $(z_1, \cdots, z_N) \in \mathbb{C}^N$ for all N and all ϕ.

From (5.7) we get the following connection between X_ϕ and its $\mathcal{H}$-transform:

$$X_\phi(\omega) = \Big[\int \tilde{X}_\phi(x + iy) d\lambda(y)\Big]_{x=\int \zeta dB} \tag{5.11}$$

Since $\tilde{X}_\phi(z)$ is not defined for all z, the integral above requires some explanation. It is defined by

$$\int \tilde{X}_\phi(x+iy)d\lambda(y) = \lim_{N\to\infty} \int \tilde{X}_\phi^{(N)}(x+iy)d\lambda(y)$$

It follows from (5.7) that this limit exists (in $L^2(\mu)$) if $x_j = \int \zeta_j dB$.

EXAMPLE 5.4 Let $X_\phi = W_\phi = \int \phi dB$ be the white noise process. If ϕ is fixed, $\rho = \|\phi\|_{L^2} \neq 0$, we may regard $\rho^{-1}\phi$ as the first function ζ_1 in our orthonormal basis $\{\zeta_k\}$. Then

$$W_\phi = \int \rho\zeta_1 dB \quad \text{so} \quad \tilde{W}_\phi(z) = \rho z_1$$

and (5.11) says that

$$\int \phi dB = [\int_{-\infty}^{\infty} \rho\cdot(x+iy)e^{-\frac{1}{2}y^2}\frac{dy}{\sqrt{2\pi}}]_{x=\int \zeta_1 dB}$$

If we don't wish to connect ϕ to $\{\zeta_k\}$ we may write

$$\phi = \sum_{k=1}^{\infty}(\phi,\zeta_k)\zeta_k \quad \text{where} \quad (\phi,\zeta_k) = \int_{\mathbb{R}} \phi(t)\zeta_k(t)dt$$

This gives

$$W_\phi = \sum_k (\phi,\zeta_k)\int \zeta_k(u)dB_u$$

and therefore

$$\tilde{W}_\phi(z) = \sum_{k=1}^{\infty}(\phi,\zeta_k)z_k \tag{5.12}$$

Hence

$$\int \tilde{W}_\phi(x+iy)d\lambda(y) = \sum_k(\phi,\zeta_k)\int_{-\infty}^{\infty}(x_k+iy_k)e^{-\frac{1}{2}y_k^2}\frac{dy_k}{\sqrt{2\pi}} = \sum_k(\phi,\zeta_k)x_k,$$

which, after the substitution $x_k = \int \zeta_k dB$ gives

$$\sum_k(\phi,\zeta_k)\int \zeta_k dB = \int (\sum_k(\phi,\zeta_k)\zeta_k(u))dB_u = \int \phi(u)dB_u.$$

The representation (5.11) is useful for explicit computations, but the main reason for the importance of the $\mathcal{H}$-transform is the following property:

THEOREM 5.5 Let X_ϕ, Y_ϕ be functional processes such that $X_\phi \diamond Y_\phi$ is a functional process. Then

$$\mathcal{H}(X_\phi \diamond Y_\phi) = \mathcal{H}(X_\phi) \cdot \mathcal{H}(Y_\phi) \tag{5.13}$$

Remark. The product on the right hand side of (5.13) is the usual complex product in the complex variables z_j but a *tensor product* in the coefficients, i.e.

$$c_\alpha(\phi)z^\alpha \cdot e_\beta(\psi)z^\beta = (c_\alpha \otimes e_\beta)(\phi \otimes \psi)z^{\alpha+\beta} = c_\alpha(\phi)e_\beta(\psi)z^{\alpha+\beta}$$

Proof. It suffices to prove this when $\alpha = (\alpha_1, \cdots, \alpha_m), \beta = (\beta_1, \cdots, \beta_k)$ and

$$X_\phi = \int \zeta^{\otimes\alpha} dB^{\otimes|\alpha|}, \quad Y_\phi = \int \zeta^{\otimes\beta} dB^{\otimes|\beta|}.$$

Then

$$X_\phi \diamond Y_\phi = \int \zeta^{\otimes\alpha} \otimes \zeta^{\otimes\beta} dB^{\otimes|\alpha+\beta|} = \int \zeta^{\otimes(\alpha+\beta)} dB^{\otimes|\alpha+\beta|}$$

and therefore

$$(X_\phi \diamond Y_\phi)^\sim(z) = z^{\alpha+\beta} = z^\alpha \cdot z^\beta = \tilde{X}_\phi(t) \cdot \tilde{Y}_\phi(t), ; \text{as claimed.}$$

COROLLARY 5.6 Suppose $X_\phi^{\diamond k}$ is a functional process for all $k \leq n$. Then for all $a_k \in \mathbb{R}, k = 1, \cdots, n$, we have

$$\sum_{k=0}^n a_k X_\phi^{\diamond k} = [\int \sum_{k=0}^n a_k \tilde{X}_\phi^k(x+iy)d\lambda(y)]_{x=\int \zeta dB}$$

§6. A functional calculus on functional processes

We now use the Hermite transform to show that for a large class of functions f and functional processes X_ϕ we can define a new functional process $f \circ X_\phi$. The key to such a functional calculus is Corollary 5.6, which can be stated as follows (under the given conditions):

$$\check{p}(X_\phi) = [\int p(\tilde{X}_\phi)(x+iy)d\lambda(y)]_{x=\int \zeta dB} \tag{6.1}$$

for every (complex) polynomial $p(z) = \sum_{k=0}^n a_k z^k$ with *real* coefficients a_k, where $\check{p}$ indicates that Wick powers of X_ϕ are used.

The idea is to extend (6.1) to a larger family of functions than the polynomials. The following result will be useful:

LEMMA 6.1 Let $f(x_1, x_2, \cdots) \in L^1(\lambda)$. Then

$$E[f(\int \zeta_1 dB, \int \zeta_2 dB, \cdots)] = \int f(x_1, x_2, \cdots) d\lambda(x)$$

Proof. It suffices to prove this in the case when f only depends on finitely many variables, and so we may assume that $f \in C_0^\infty(\mathbb{R}^n)$ for some n. Then f is the inverse Fourier transform of its Fourier transform $\hat{f}$:

$$f(x_1, \cdots, x_n) = (2\pi)^{-\frac{n}{2}} \cdot \int_{\mathbb{R}^n} \hat{f}(y_1, \cdots, y_n) \cdot e^{i(x,y)} dy$$

Hence, by (2.1)

$$E[f(\int \zeta_1 dB, \cdots, \int \zeta_n dB)] = (2\pi)^{-\frac{n}{2}} \cdot \int_{\mathbb{R}^n} \hat{f}(y_1, \cdots, y_n) E[e^{i<\omega, \sum y_j \zeta_j>}] dy$$

$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{f}(y_1, \cdots, y_n) e^{-\frac{1}{2} \sum y_j^2} dy = (2\pi)^{-n} \int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} f(x_1, \cdots, x_n) e^{-i(x,y)} dx) e^{-\frac{1}{2}|y|^2} dy$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} f(x_1, \cdots, x_n) \cdot (\int_{\mathbb{R}^n} e^{-i(x,y) - \frac{1}{2}|y|^2} dy) dx$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} f(x_1, \cdots, x_n) \cdot (2\pi)^{\frac{n}{2}} \cdot e^{-\frac{1}{2}|x|^2} dx = \int f(x) d\lambda(x), \text{ as claimed.}$$

Here we have used the well-known formula

$$\int_{-\infty}^{\infty} e^{i\alpha t - \beta t^2} dt = (\frac{\pi}{\beta})^{\frac{1}{2}} \cdot e^{-\frac{\alpha^2}{4\beta}} \tag{6.2}$$

COROLLARY 6.2 Let $g : \mathbb{R} \to \mathbb{R}$ be measurable and such that $g(X_\phi) \in L^1(\mu)$. Then

$$E[g(X_\phi)] = \int g(\int \tilde{X}_\phi(x + iy) d\lambda(y)) d\lambda(x).$$

In particular,

$$E[|X_\phi|^2] \le \int \int |\tilde{X}_\phi|^2 (x + iy) d\lambda(x) d\lambda(y)$$

DEFINITION 6.3 a) We let $A^2 = A^2(\lambda \times \lambda)$ be the set of all formal power series

$$f(z_1, z_2, \cdots) = \sum_{n=0}^{\infty} (\sum_{|\alpha|=n} c_\alpha z^\alpha)$$

with real coefficients c_α for all $\alpha = (\alpha_1, \cdots, \alpha_m)$, such that

$$f^{(N)}(z) := f(z^{(N)}) \text{ is analytic for each } N$$

(using the notation of (5.10)) and $f \in L^2(\lambda \times \lambda)$, i.e.

(6.3)
$$\|f\|^2_{L^2(\lambda\times\lambda)} := \int\int |f(x+iy)|^2 d\lambda(x) d\lambda(y) := \lim_{N\to\infty} \int\int |f^{(N)}(x+iy)|^2 d\lambda(x) d\lambda(y) < \infty$$

b) Define $\mathcal{G}$ to be the set of all functional processes X_ϕ such that $\tilde{X}_\phi^k \in A^2(\lambda \times \lambda)$ for all $k = 1, 2, 3, \cdots$

Remark:

$$\mathcal{G} \text{ is an algebra (under Wick multiplication).}$$

To prove this it suffices to show that if $X_\phi \in \mathcal{G}$, then $X_\phi^{\diamond 2} \in \mathcal{G}$ also. And this follows from Theorem 5.4.

Our first main result in this section is the following:

THEOREM 6.4 (Functional calculus)
Let $X_\phi \in \mathcal{G}$ and $f : \mathbb{C} \to \mathbb{C}$ be given and assume that there exist polynomials $p_k : \mathbb{C} \to \mathbb{C}$ with real coefficients such that

$$p_k(\tilde{X}_\phi) \to f(\tilde{X}_\phi) \text{ in } L^2(\lambda \times \lambda) \text{ as } k \to \infty. \tag{6.4}$$

Then

$$\check{f}(X_\phi) := [\int f(\tilde{X}_\phi)(x+iy) d\lambda(y)]_{x=\int \zeta dB} \tag{6.5}$$

defines a functional process.

Proof. For each k we have that

$$\check{p}_k(X_\phi) = [\int p_k(\tilde{X}_\phi)(x+iy) d\lambda(y)]_{x=\int \zeta dB}$$

is a functional process and, by Lemma 6.1,

$$E[|\check{p}_k(X_\phi) - \check{p}_l(X_\phi)|^2] = E[|\int (p_k(\tilde{X}_\phi) - p_l(\tilde{X}_\phi)) \circ (\int \zeta dB + iy) d\lambda(y)|^2]$$
$$\le E[\int |p_k(\tilde{X}_\phi) - p_l(\tilde{X}_\phi)|^2 \circ (\int \zeta dB + iy) d\lambda(y)] = \|p_k(\tilde{X}_\phi) - p_l(\tilde{X}_\phi)\|_{L^2(\lambda\times\lambda)}$$

$\to 0$ as $k, l \to \infty$.

We conclude that $\{\breve{p}_k(X_\phi)\}$ constitute a Cauchy sequence in $L^2(\mu)$. If we write

$$\breve{p}_k(X_\phi) = \sum_n \int F_\phi^{(n,k)} dB^{\otimes n} \quad \text{then}$$

$$E[|\breve{p}_k(X_\phi) - \breve{p}_l(X_\phi)|^2] = \sum_n n! \|F_\phi^{(n,k)} - F_\phi^{(n,l)}\|^2_{L^2(\mathbb{R}^n)},$$

so for all fixed n we see that $\{F_\phi^{(n,k)}\}_{k=1}^\infty$ converges to a limit $G_\phi^{(n)}$, say, in $L^2(\mathbb{R})$. Hence

$$\lim \breve{p}_k(X_\phi) = \sum_n \int G_\phi^{(n)} dB^{\otimes n},$$

which is a functional process.

EXAMPLE 6.5 *The exponential of white noise* is defined by

$$\text{(6.6)} \qquad \operatorname{Exp}(W_t) := \sum_{n=0}^\infty \frac{1}{n!} W_t^{\odot n} := [\int \exp(\tilde{W}_t)(x+iy)d\lambda(y)]_{x=\int \zeta dB}$$

With $\phi \in \mathcal{S}$ given we can choose $\zeta_1 = \rho^{-1}\phi$ where $\rho = \|\phi\|_{L^2}$ and this gives (see Ex. 5.3)

$$\tilde{W}_t(z) = \rho z_1 \in L^p(\lambda \times \lambda) \quad \text{for all } p < \infty$$

Hence $W_t \in \mathcal{G}$ and clearly

$$\exp(\tilde{W}_t)(z) = \exp(\rho z_1) = \lim_{k \to \infty} \sum_{n=0}^k \frac{1}{n!}(\rho z_1)^n = \lim_{k \to \infty} \sum_{n=0}^k \frac{1}{n!}(\tilde{W}_t)^n(z),$$

the limit being taken in $L^2(\lambda \times \lambda)$. So Theorem 6.4 applies to define (6.6) as a functional process.

EXAMPLE 6.6 The square of white noise is given by

$$Y_\phi = W_\phi^{\diamond 2} \doteq \int_{\mathbb{R}^2} \phi \otimes \phi dB^{\otimes 2},$$

so, with $\phi = \rho\zeta_1, \tilde{Y}_\phi(z) = \rho^2 z_1^2$ and hence $Y_\phi \in \mathcal{G}$. However,

$$\exp(\tilde{Y}_\phi) = e^{\rho^2 z_1^2} \text{ which belongs to } L^2(\lambda \times \lambda) \text{ only for } \rho < \frac{1}{2},$$

but for such ϕ we see that $\operatorname{Exp}(W_\phi^{\diamond 2})$ is well-defined.

The inverse $\mathcal{H}$-transform

The $\mathcal{H}$-transform assigns to each functional process X_ϕ a function $\tilde{X}_\phi : \mathbb{C}_0^{\mathbb{N}} \to \mathbb{C}$. Conversely, starting with a function $g : \mathbb{C}_0^{\mathbb{N}} \to \mathbb{C}$ it is useful to be able to find a functional process X whose $\mathcal{H}$-transform is g. Clearly the candidate for X is

$$\text{(6.7)} \qquad X = [\int g(z) d\lambda(y)]_{x=\int \zeta dB}$$

but the question is for what g this makes sense and defines a functional process.

DEFINITION 6.7
Suppose that we are given distributions
(6.8) $c_0(\cdot) \in H^{-\infty}(\mathbb{R})$ and
(6.9) $c_\alpha(\cdot) \in H^{-\infty}(\mathbb{R}^n)$ for each multiindex α with $|\alpha| = n \geq 1$, such that
(6.10) $\sum\limits_{n=0}^{\infty} n! \sum\limits_{|\alpha|=n} c_\alpha^2(\phi) < \infty$ for all $\phi \in \mathcal{S}$.

Then, as shown by the proof of Lemma 5.3, the function

$$g_\phi(z) := \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha(\phi) z^n \tag{6.11}$$

is defined and analytic on $\mathbb{C}_0^{\mathbb{N}}$, for each $\phi \in \mathcal{S}$. The set of all such functions g_ϕ is denoted by $\mathcal{P}$.

THEOREM 6.8 (Inverse $\mathcal{H}$-transform)
Let $g_\phi \in \mathcal{P}$. Then

$$X_\phi := [\int g_\phi(z) d\lambda(y)]_{x=\int \zeta dB}$$

defines a functional process whose $\mathcal{H}$-transform is g_ϕ.

We will use $\mathcal{H}^{-1}(g_\phi)$ or $\breve{g}_\phi$ as notation for the inverse Hermite transform. (Strictly speaking this notation is slightly in conflict with the notation of (6.5), but this should not cause any difficulties).

Proof of Theorem 6.8. If $g_\phi(z) = \sum\limits_n \sum\limits_{|\alpha|=n} c_\alpha(\phi) z^\alpha$ then by (5.5)

$$\int g_\phi(z) d\lambda(y) = \sum_n \sum_{|\alpha|=n} c_\alpha(\phi) h_\alpha(x),$$

where

$$h_\alpha(x) = h_{\alpha_1}(x_1) h_{\alpha_2}(x_2) \cdots h_{\alpha_m}(x_m) \text{ if } \alpha = (\alpha_1, \cdots, \alpha_m)$$

So by (5.4) we get

$$X_\phi := [\int g_\phi(z) d\lambda(y)]_{x=\int \zeta dB} = \sum_n \sum_{|\alpha|=n} c_\alpha(\phi) \int \zeta^{\otimes \alpha} dB^{\otimes n}$$

Since $E[|X_\phi|^2] = \sum\limits_n n! \sum\limits_{|\alpha|=n} c_\alpha^2(\phi) < \infty$, we conclude that X_ϕ is a functional process.

§7. Positive noise

We now return to the concept of *positive noise*, mentioned in the introduction:

DEFINITION 7.1. A functional process

$$X_t(\omega) = \sum_{n=0}^{\infty} \int F_t^{(n)}(u) dB_u^{\otimes n}(\omega)$$

is called a *positive noise* if

$$X_\phi(\omega) \geq 0 \text{ a.s. for all } \phi \in \mathcal{S} \text{ with } \|\phi\|_{L^2} \text{ sufficiently small.} \tag{7.1}$$

EXAMPLE 7.2.

$$W_t^{\diamond 2}(\omega) = \int \delta_t^{\otimes 2} dB^{\otimes 2}$$

is *not* a positive noise, since

$$W_\psi^{\diamond 2}(\omega) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{v} \psi(u,v) dB_u(\omega) dB_v(\omega) \text{ for all } \psi \in L^2(\mathbb{R}^2),$$

so choosing e.g. $\psi(u,v) = \chi_{[0,1]\times[0,1]}(u,v)$ we get

$$W_\psi^{\diamond 2}(\omega) = 2 \int_0^1 \Big(\int_0^v dB_u(\omega) \Big) dB_u(\omega) = 2 \int_0^1 B_v(\omega) dB_v(\omega) = B_1^2(\omega) - 1,$$

which is not positive a.s.

However, we have the following:

EXAMPLE 7.3 The exponential of white noise,

$$N_t := \operatorname{Exp}(W_t) = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\int \delta_t^{\otimes n} dB^{\otimes n} \Big)$$

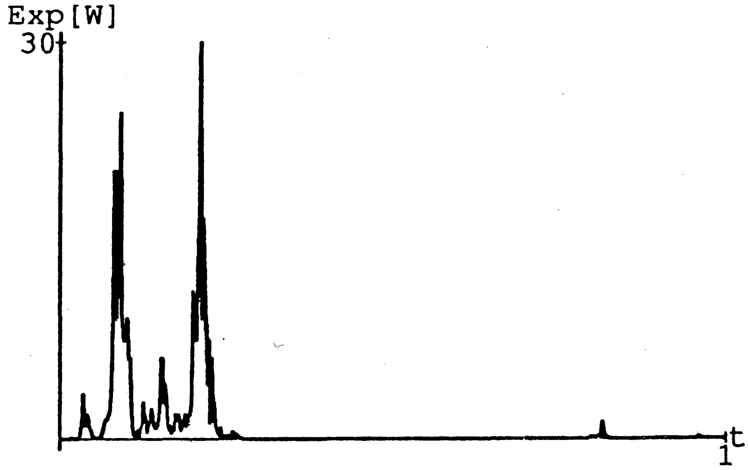
is a positive noise.

To see this we use the computation from Example 6.5:

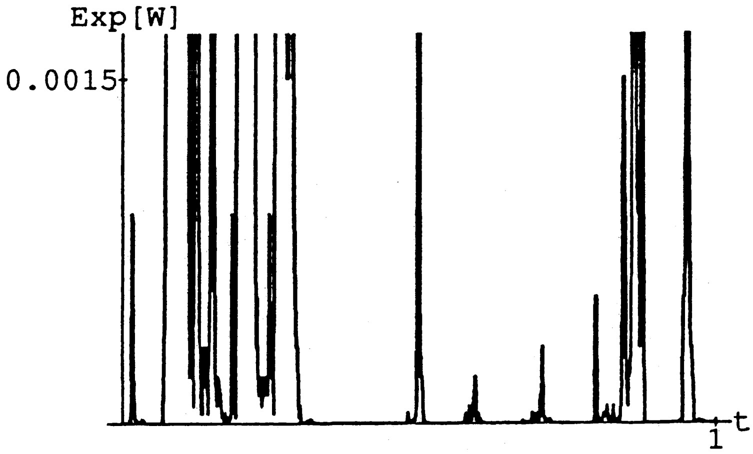
$$\begin{aligned} N_\phi(\omega) &= \Big[\int_{-\infty}^{\infty} \exp(\rho x_1 + i\rho y_1) \cdot e^{-\frac{1}{2}y_1^2} \frac{dy_1}{\sqrt{2\pi}} \Big]_{x_1 = \int \zeta_1 dB} \\ &= \exp\Big(\int \phi dB \Big) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\rho y - \frac{1}{2}y^2} dy, \end{aligned}$$

Figure 1a,b,c
The exponential $Exp[W_\phi]$ of white noise W_ϕ.

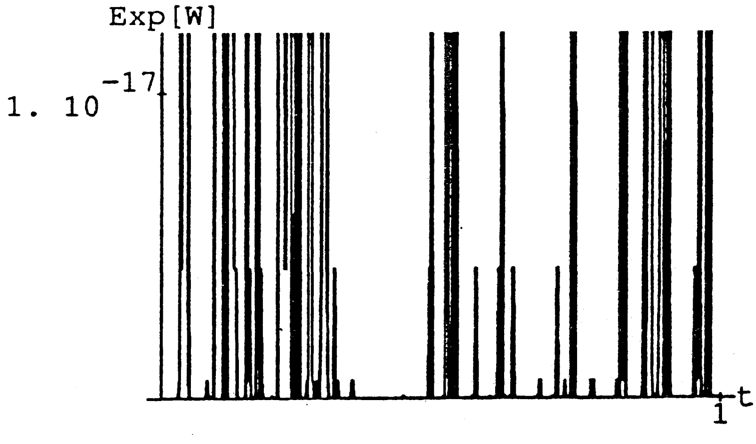
a) A path of $Exp[W_\phi]$ showing all values. Suppϕ in [t,t+1/10]



b) A path of $Exp[W_\phi]$ showing "fine structure".
Largest value was 7.0 . Suppϕ in [t,t+1/25].



c) A path of $Exp[W_\phi]$ showing "micro structure".
Suppϕ in [t,t+1/100]. The graph is now torn completely apart.



which combined with (6.2) gives

$$\mathrm{Exp}(W_\phi) = \exp\Big(\int \phi dB - \frac{1}{2}\|\phi\|_{L^2}^2\Big); \; \phi \in \mathcal{S}. \tag{7.2}$$

In particular, $\mathrm{Exp}(W_\phi) \geq 0$ a.s.

Computer simulations of $\mathrm{Exp}[W_\phi]$ for 3 choices of "averages" ϕ_1, ϕ_2, ϕ_3 with supports $[t, t+\frac{1}{10}], [t, t+\frac{1}{25}]$ and $[t, t+\frac{1}{100}]$, respectively are shown on Figure 1 (In all cases $\|\phi_j\|_{L^1} = 1$).

Returning to the examples (1.2) and (1.3) in the introduction, we see that a crucial question for the usefulness of the concept of positive functional processes is the following:

If X_ϕ, Y_ϕ are two positive functional processes and $X_\phi \diamond Y_\phi$ is defined, is $X_\phi \diamond Y_\phi$ also a positive functional process?

The main result in this section is an affirmative answer to this question, together with a characterization of positiveness for functional processes in terms of positive definiteness of its Hermite transform. This characterization is analogous to Theorem 4.1 in Potthoff (1987), but there the setting, the transform and the methods are different.

THEOREM 7.4 Let X be a functional process and fix $\phi \in \mathcal{S}$. Then $X_\phi(\omega) \geq 0$ a.s. if and only if

$$g_n(y) = \tilde{X}_\phi^{(n)}(iy)e^{-\frac{1}{2}y^2}; y \in \mathbb{R}_0^{\mathbb{N}} \tag{7.3}$$

is *positive definite* for all n, where $\tilde{X}_\phi^{(n)}(z)$ is defined by (5.10).

Remark: Condition (7.3) means the following:

For all positive integers m and all $y^{(1)}, \cdots, y^{(m)} \in \mathbb{R}_0^{\mathbb{N}}, a = (a_j) \in \mathbb{C}_0^{\mathbb{N}}$ we have

$$\sum_{j,k}^{m} a_j \overline{a}_k g_n(y^{(j)} - y^{(k)}) \geq 0. \tag{7.4}$$

Proof. Recall that $F(z) := F_n(z_1, \cdots, z_n) := \tilde{X}_\phi^{(n)}(z)$ is analytic in each of the variables $z_1, \cdots, z_n$.

Let

$$H_n(x) = H_n(x_1, \cdots, x_n) = \int \tilde{X}_\phi^{(n)}(x+iy)d\lambda(y) = \int F(x+iy)e^{-\frac{1}{2}y^2}(2\pi)^{-\frac{n}{2}}dy \tag{7.5}$$

where $y = (y_1, \cdots, y_n), dy = dy_1 \cdots dy_n$. We write this as

$$e^{-\frac{1}{2}x^2} \cdot \int F(z)e^{\frac{1}{2}z^2} \cdot e^{-ixy}(2\pi)^{-\frac{n}{2}}dy = e^{-\frac{1}{2}x^2}(2\pi)^{-\frac{n}{2}} \int G(z)e^{-ixy}dy, \tag{7.6}$$

where $z=(z_1,\cdots,z_n), z^2=z_1^2+\cdots+z_n^2, xy=\sum_{j=1}^{n}x_jy_j$ and $G(z)=G_n(z)=F(z)e^{\frac{1}{2}z^2}$ is analytic.

Consider the function

$$f(x;\xi)=\int G(x+iy)e^{-i\xi y}dy;x,\xi\in\mathbb{R}^n \tag{7.7}$$

By the Cauchy-Riemann equations we have

$$\frac{\partial f}{\partial x_1}=\int\frac{\partial G}{\partial x_1}\cdot e^{-i\xi y}dy=\int(-i)\frac{\partial G}{\partial y_1}e^{-i\xi y}dy$$

Now

$$\int_{-\infty}^{\infty}(-i)\frac{\partial G}{\partial y_1}\cdot e^{-i\xi_1 y_1}dy_1=i\int_{-\infty}^{\infty}G(z)e^{i\xi_1 y_1}(-i\xi_1)dy_1$$

and this gives

$$\frac{\partial f}{\partial x_1}=\xi_1 f(x;\xi)$$

Hence

$$f(x_1,x_2,\cdots,x_n;\xi)=f(0,x_2,\cdots,x_n;\xi)e^{\xi_1x_1}$$

and similarly for $x_2,\cdots,x_n$. Therefore

$$f(x;\xi)=f(0;\xi)e^{\xi x}=e^{\xi x}\int G(iy)e^{-i\xi y}dy \tag{7.8}$$

We conclude from (7.5)-(7.8) that

$$\begin{aligned}H_n(x)=\int\tilde{X}_\phi^{(n)}(x+iy)d\lambda(y)&=e^{\frac{1}{2}x^2}(2\pi)^{-\frac{n}{2}}\int\tilde{X}_\phi^{(n)}(iy)e^{-\frac{1}{2}y^2}e^{-ixy}dy\\&=e^{\frac{1}{2}x^2}\hat{g}_n(x),\end{aligned} \tag{7.9}$$

where $\hat{g}_n(\xi)=(2\pi)^{-n/2}\int g_n(y)e^{-i\xi y}dy$ is the Fourier transform of g_n.

Note that $g_n\in\mathcal{S}$ and hence $\hat{g}_n\in\mathcal{S}$. Therefore we can apply the Fourier inversion to obtain

$$\begin{aligned}g_n(\xi)&=(2\pi)^{-\frac{n}{2}}\int\hat{g}_n(-x)e^{ix\xi}dx\\&=\int e^{i\xi x}H_n(x)d\lambda(x),\xi\in\mathbb{R}^n.\end{aligned} \tag{7.10}$$

Hence, if $\xi^{(1)},\xi^{(2)}\cdots\in\mathbb{R}_0^{\mathbb{N}}$ and $a=(a_j)\in\mathbb{C}_0^{\mathbb{N}}$ we have

$$\sum_{j,k}^{M}a_j\overline{a}_kg_n(\xi^{(j)}-\xi^{(k)})=\int|\gamma(x)|^2H_n(x)d\lambda(x),$$

where $\gamma(x)$ is the vector $[a_j e^{i\xi^{(j)}x}]_{1\leq j\leq m}$.

We conclude that

$$\sum_{j,k}^{m} a_j\overline{a}_k g_n(\xi^{(j)}-\xi^{(k)}) = E[|\gamma(\int \zeta dB)|^2(\int \tilde{X}_\phi^{(n)}(\int \zeta dB + iy)d\lambda(y))] \tag{7.11}$$
$$\to E[|\gamma(\int \zeta dB)|^2 X_\phi(\omega)] \text{ as } n\to\infty.$$

So if $X_\phi(\omega)\geq 0$ for $a.a.\omega$ we deduce that

$$\lim_{n\to\infty}\sum_{j,k}^{m} a_j\overline{a}_k g_n(\xi^{(j)}-\xi^{(k)})\geq 0 \tag{7.12}$$

But with $\xi^{(1)},\cdots\xi^{(m)}\in\mathbb{R}_0^{\mathbb{N}}$ fixed $g_n(\xi^{(j)}-\xi^{(k)})$ becomes eventually constant as $n\to\infty$ so (7.12) is equivalent to (7.4).

Conversely, if g_n is positive definite then $H_n(x)\geq 0$ for $\eta - a.a.x$ and if this holds for all n we have $X_\phi(\omega)\geq 0$ a.s. by (7.5).

COROLLARY 7.5 Let X, Y be two functional processes such that $X\diamond Y$ is a functional process. If $X\geq 0$ and $Y\geq 0$ then $X\diamond Y\geq 0$.

Proof. For $\phi\in\mathcal{S}$ consider $\tilde{X}_\phi^{(n)}(iy)e^{-\frac{1}{2}y^2}$ as before and similarly $\tilde{Y}_\phi^{(n)}(iy)e^{-\frac{1}{2}y^2}$. Replacing ϕ by $\rho\phi$ where $\rho>0$ we obtain from Theorem 7.4 that

$$g_n^{(\rho)}(y) := \tilde{X}_\phi(i\rho y)e^{-\frac{1}{2}y^2} \text{ is positive definite,}$$

hence

$$\sigma_n(y) := \tilde{X}_\phi^{(n)}(iy)e^{-\frac{1}{2}(\frac{y}{\rho})^2} \text{ is positive definite,} \tag{7.13}$$

and similarly

$$\gamma_n(y) := \tilde{Y}_\phi^{(n)}(iy)e^{-\frac{1}{2}(\frac{y}{\rho})^2} \text{ is positive definite.} \tag{7.14}$$

Therefore the product $\sigma_n\gamma_n(y) = (\tilde{X}_\phi^{(n)}\cdot\tilde{Y}_\phi^{(n)})(iy)e^{-(\frac{y}{\rho})^2}$ is positive definite.

Choosing $\rho=\sqrt{2}$ this gives that

$$(X\diamond Y)_\phi^{\sim(n)}(iy)e^{-\frac{1}{2}y^2} \text{ is positive definite,}$$

so from Theorem 7.4 we have $X\diamond Y\geq 0$.

EXAMPLE 7.6 To illustrate Theorem 7.4 let us check that the exponential of white noise, $X_t = \mathrm{Exp}(W_t)$, is positive: Choosing $\phi = \rho\zeta$, we have

$$\tilde{X}_t(z) = \exp(\rho z_1) \text{ so that}$$
$$g(y) = \exp(i\rho y_1 - \frac{1}{2}y^2).$$

Now

$$\sum a_j \bar{a}_k g(y^{(j)} - y^{(k)}) = \sum a_j \bar{a}_k e^{i\rho y_1^{(j)}} \cdot e^{-i\rho y_1^{(k)}} \cdot e^{-\frac{1}{2}(y^{(j)}-y^{(k)})^2}$$
$$= \sum (a_j e^{i\rho y_1^{(j)}})(\overline{a_k e^{i\rho y_1^{(k)}}}) e^{-\frac{1}{2}(y^{(j)}-y^{(k)})^2} \geq 0,$$

since $f(y) = e^{-\frac{1}{2}y^2}$ is positive definite. Hence g is positive definite and therefore $X_t \geq 0$.

EXAMPLE 7.7 Returning to equation (1.2) in the introduction, we see that Corollary 7.5 is necessary for our concept of a positive noise to make sense in such a model: If for example $r = 0, \alpha = 1$ so that

$$\frac{dX_t}{dt} = N_t \diamond X_t, X_0 = 1$$

with N_t a positive noise, then Corollary 7.5 gives that $\frac{dX_t}{dt}$ is indeed positive if X_t is.

§8. The solution of stochastic differential equations involving functionals of white noise.

As pointed out in the introduction it is important that our generalized noise concept leads to stochastic differential equations which can be handled mathematically. In this section we show that this is indeed the case. The main idea is to transform the original noise equation into a (deterministic) differential equation for the Hermite transform. By applying the inverse Hermite transform to the solution of this deterministic equation we will - under some conditions - obtain a functional process which solves the original equation.

To illustrate this method we look at the 1-dimensional version of equation (1.3), i.e.

$$(k(t)X_t')' = 0 \tag{8.1}$$

where

$$k(t) = \mathrm{Exp}(\varepsilon W_t), \varepsilon > 0 \text{ constant},$$

is our model for the positive noise $k(t)$, and the product is interpreted as a Wick product $\diamond$. Taking $\mathcal{H}$-transform we get the equation

$$\exp(\varepsilon \tilde{W}_t) \cdot \tilde{X}_t' = C_1 \ (C_1 \text{ constant})$$

From (5.12) we have

$$\tilde{W}_\phi(z) = \sum_j (\phi, \zeta_j) z_j; \ \phi \in \mathcal{S}$$

So

$$\tilde{X}'_\phi(z) = C_1 \exp(-\varepsilon \sum_j (\phi, \zeta_j) z_j)$$

which gives

$$X'_\phi(z) = C_1 \cdot \lim_{k\to\infty} [\int \exp(-\varepsilon \sum_{j=1}^{k} (\phi, \zeta_j) z_j) d\lambda(y)]_{x=\int \zeta dB}$$

$$(\text{by formula 6.2}) = C_1 \lim_{k\to\infty} [\exp(-\varepsilon \sum_{j=1}^{k} ((\phi, \zeta_j) x_j + \frac{\varepsilon^2}{2} (\phi, \zeta_j)^2))]_{x=\int \zeta dB}$$

$$= C_1 \cdot \exp(-\varepsilon \int \phi dB - \frac{\varepsilon^2}{2} \|\phi\|^2_{L^2})$$

In other words,

$$X'_\phi = C_1 \mathrm{Exp}(-\varepsilon W_\phi) = C_1 \sum_{n=0}^{\infty} \frac{(-1)^n \varepsilon^n}{n!} \int \delta_t^{\otimes n} dB^{\otimes n}, \tag{8.2}$$

as we would have obtained by direct formal computation from (8.1).

Note that if we define $G : \mathcal{S}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by

$$< G, \phi \otimes \psi >= \frac{1}{2} \int_0^\infty \{\phi(u+s)\psi(v+s) + \phi(v+s)\psi(u+s)\} ds \cdot \chi_{\{u>0, v>0\}} \tag{8.3}$$

then if supp $\phi \subset \mathbb{R}^+$, supp $\psi \subset \mathbb{R}^+$ we have

$$< \frac{\partial G}{\partial x_1}, \phi \otimes \psi >= - < G, \phi' \otimes \psi >$$

$$= -\frac{1}{2} \int_0^\infty \{\phi'(u+s)\psi(v+s) + \phi'(v+s)\psi(u+s)\} ds,$$

so

$$< \frac{\partial G}{\partial x_1}, \phi \otimes \phi >= -\frac{1}{2} \int_0^\infty \frac{d}{ds} (\phi(u+s)\phi(v+s)) ds = -\frac{1}{2} \phi(u)\phi(v)$$

Hence

$$\frac{d}{dt} G_t = (\frac{\partial G}{\partial x_1} + \frac{\partial G}{\partial x_2})_{x=(t,t)} = \delta_t \otimes \delta_t \text{ for } t > 0. \tag{8.4}$$

Similarly we see that if we more generally define

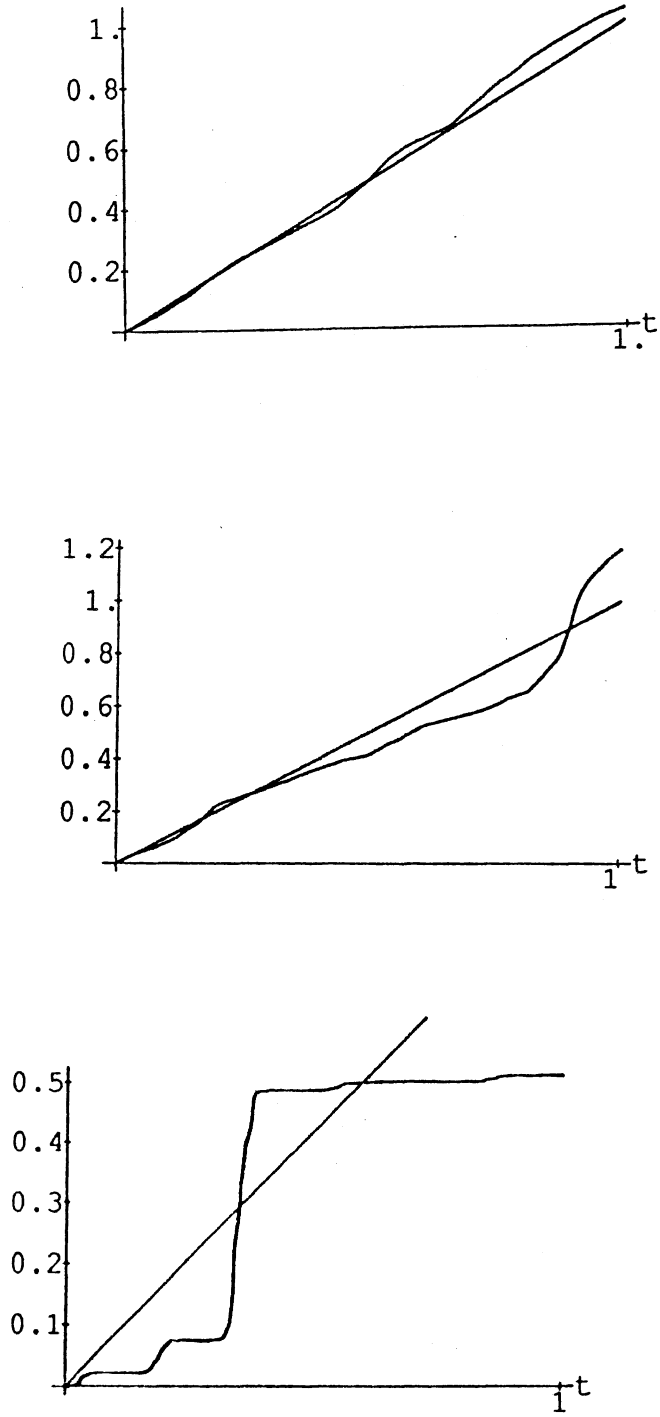
$$G_\phi^{(n)} :=< G^{(n)}, \phi^{\otimes n} >:= \int_0^\infty \phi(u_1+s)\phi(u_2+s) \cdots \phi(u_n+s) ds \cdot \chi_{\{u_i>0, \forall i\}} \tag{8.5}$$

Figure 2

ε=0.1

ε=0.2

ε=1.0



then

$$\frac{d}{d\phi}G_\phi^{(n)} = \phi^{\otimes n} \text{ if supp } \phi \subset \mathbb{R}^+$$

In other words

(8.6) $$\frac{d}{dt}G_t^{(n)} = \delta_t^{\otimes n} \text{ for } t > 0.$$

Using this in (8.2) we conclude that

(8.7) $$X_t = X_t^{(\varepsilon)} = C_2 + C_1 t + C_1 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon^n}{n!} \int G_t^{(n)}(u) dB_u^{\otimes n} \ (C_2 \text{ constant})$$

is the solution of (8.1).

Computer simulations of this solution with $C_1 = 1$ and $C_2 = 0$ and various choices of ε are shown on Figure 2. There we have chosen $\phi(u) = \phi_t(u) = \frac{1}{h} \cdot \chi_{[t,t+h]}(u)$ with $h = 0.1$, so X_t really means X_{ϕ_t}.

The following question is of interest:
How does the stochastic solution $X_t^{(\varepsilon)}$ differ from the no-noise solution $X_t^{(0)} = C_2 + C_1 t$? How big error do we make if we replace the stochastic permeability $k(t) = \text{Exp}(\varepsilon W_t)$ by its average $k_0(t) = 1$? The answer depends heavily on the ratio between ε^2 and the support h of the averaging function $\phi(u) = \frac{1}{h} \cdot \chi_{[t,t+h]}(u)$:

THEOREM 8.1 For $\varepsilon > 0$ let

(8.8) $$X_t^{(\varepsilon)} = t + \sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon^n}{n!} \int G_t^{(n)}(u) dB_u^{\otimes n}$$

be the solution of (8.1) with $C_2 = X_0^{(\varepsilon)} = 0, C_1 = (X^{(\varepsilon)})'_0 = 1$. Then

(8.9) $$E[X_t^{(\varepsilon)}] = t$$

and

(8.10) $$E[(X_t^{(\varepsilon)} - t)^2] \leq (t+h)[\exp(\frac{\varepsilon^2}{h}) - 1]$$

Proof. (8.9) is straightforward from (8.8). Consider

(8.11) $$\begin{aligned} E[(X_t^{(\varepsilon)} - t)^2] &= E[(\sum_{n=1}^{\infty} \frac{(-1)^n \varepsilon^n}{n!} \int G_t^{(n)}(u) dB_u^{\otimes n})^2] \\ &= \sum_{n=1}^{\infty} \frac{\varepsilon^{2n}}{n!} \int (G_t^{(n)}(u))^2 du \end{aligned}$$

Now

$$\phi(u_1 + s) \cdots \phi(u_n + s) \neq 0 \text{ only if } u_i + s \in [t, t+h] \text{ for all } i.$$

Hence

$$\int (G_t^{(n)}(u))^2 du \leq \int_{(\mathbb{R}+)^n} \int_0^{t+h} h^{-2n} \chi_{\{u_i+s\in[t,t+h],\forall i\}} ds du$$

$$= h^{-2n} \int_0^{t+h} (\prod_{i=1}^n \int_0^\infty \chi_{\{u_i\in[t-s,t-s+h]\}} du_i) ds$$

$$\leq h^{-2n} \int_0^{t+h} h^n ds = (t+h)h^{-n}.$$

Substituted in (8.11) this gives (8.10).

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